Hot Air Balloon

At the West Texas Balloon Festival, a hot air balloon is sighted at an altitude of 800 feet. It appears to be descending at a steady rate of 20 feet per minute. Spectators wonder how the altitude of the balloon is changing as time passes.

1. Write a function rule (equation) to represent the relationship between the variables in this scenario.

2. How high was the balloon 5 minutes before it was sighted? Make a table of values and/or a graph that could be used to answer this question.

3. How long will it take the balloon to descend to an altitude of 20 feet? How long will it take the balloon to land?

4. At the instant the first balloon is sighted, a second balloon is also observed at an altitude of 1,200 feet and descending at a rate of 20 feet per minute. How do the descent and landing time of the second balloon compare with that of the first balloon? What does this mean graphically?

5. At the instant the first balloon is sighted, a third balloon is also observed at an altitude of 800 feet and descending at a rate of 30 feet per minute. How do the descent and landing time of the third balloon compare with that of the first balloon? What does this mean graphically?

6. At the instant the first balloon is sighted, a fourth balloon is launched from the ground. It rises at a rate of 30 feet per minute. When will the first and fourth balloons be at the same altitude? What is that altitude? What does this mean graphically?

Notes

CCSS Content Task

(7.EE) **Solve real-life and mathematical problems using numerical and algebraic expressions and equations.**

4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

7. Solve linear equations in one variable.

b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

8. Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

Scaffolding Questions

- What are the variables in this scenario?
- Which quantity is the dependent variable? The independent variable?
- What kind of function models the situation? How do you know?
- What decisions must you make to build a table for the function?
- What decisions must you make to graph the function?
- How can you determine the balloon's height at any given time?
- How can you determine the time it takes the balloon to reach a given height?
- How does a change in initial altitude affect the equation? The graph? The table?
- How does a change in the rate of ascent (or descent) affect the equation? The graph? The table?

Sample Solutions

1. Write a function rule (equation) to represent the relationship between the variables in this scenario.

   Possible function rules: The starting height, 800 feet, decreases at a rate of 20 feet per minute. The height (h) equals 800 minus the product of 20 and the number of minutes (m): $h = 800 - 20m$. Or, if students are using a graphing calculator, they can use y to represent height and x to represent the number of minutes: $y = 800 - 20x$.

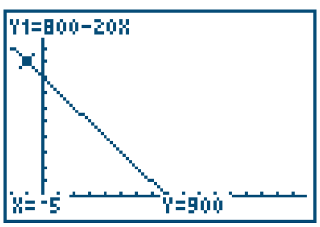
2. How high was the balloon 5 minutes before it was sighted? Make a table of values and/or a graph that could be used to answer this question.

   The value of y is 900 when m is -5. Therefore, the balloon was at 900 feet 5 minutes before it was first sighted.

Possible table:

m	$800 - 20m$	h
–5	800 – 20(–5)	900
0	800 – 20(0)	800
5	800 – 20(5)	700
10	800 – 20(10)	600
15	800 – 20(15)	500
20	800 – 20(20)	400
25	800 – 20(25)	300
30	800 – 20(30)	200
35	800 – 20(35)	100
40	800 – 20(40)	0

A graph can also be used to examine the situation:



3. How long will it take the balloon to descend to an altitude of 20 feet? How long will it take the balloon to land?

To discover how long it will take the balloon to descend to an altitude of 20 feet, solve for m:

$$\begin{aligned} 800 - 20m &= 20 \\ -20m &= -780 \\ m &= 39 \end{aligned}$$

It will take the balloon 39 minutes to descend to 20 feet above the ground.

Solve $800 - 20m = 0$ for m to get $m = 40$. It will take the balloon 40 minutes to land.

A graph or table can also be examined to determine when the height is 0.

b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(8.F) **Define, evaluate, and compare functions.**

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

(A-CED) **Create equations that describe numbers or relationships**

1. Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(A-REI) **Solve equations and inequalities in one variable**

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

(A-REI) **Solve systems of equations**

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(F-IF) **Understand the concept of a function and use function notation**

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

(F-IF) **Analyze functions using different representations**

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

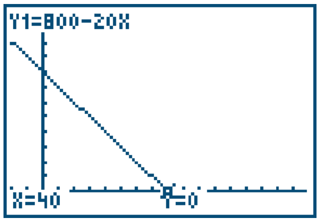
(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

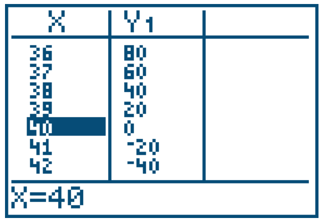
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

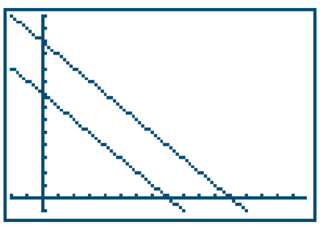




4. At the instant the first balloon is sighted, a second balloon is also observed at an altitude of 1,200 feet and descending at a rate of 20 feet per minute. How do the descent and landing time of the second balloon compare with that of the first balloon? What does this mean graphically?

   The second balloon is at a higher altitude than the first balloon but is descending at the same rate. The second balloon will take longer to land. The function rule for the second balloon is $y = 1{,}200 - 20x$. The second balloon will land in 60 minutes, or 20 minutes after the first balloon.

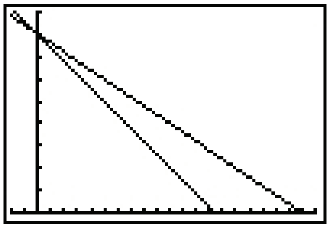
   The graph for this balloon has different y- and x-intercepts than the graph for the first balloon. The graphs for both balloons are parallel lines because they have the same slope.



5. At the instant the first balloon is sighted, a third balloon is also observed at an altitude of 800 feet and descending at a rate of 30 feet per minute. How do the descent and landing time of the third balloon compare with that of the first balloon? What does this mean graphically?

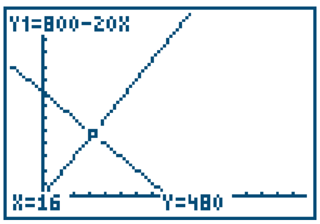
   The third balloon starts at the same height as the first but descends faster. Therefore, the third balloon will land sooner. The function rule describing this balloon is $y = 800 - 30m$. The third balloon lands in about 27 minutes, or about 13 minutes before the first balloon.

   The graph for this balloon has the same y-intercept as the first balloon's graph but a different x-intercept. The x-intercept for the third balloon is less than the x-intercept for the first balloon. The graph of the third balloon's descent is steeper than that of the first balloon's descent.



6. At the instant the first balloon is sighted, a fourth balloon is launched from the ground. It rises at a rate of 30 feet per minute. When will the first and fourth balloons be at the same altitude? What is that altitude? What does this mean graphically?

   The function rule describing the fourth balloon is $y = 30x$. To see if the first and fourth balloons are ever at the same altitude, explore with tables or graphs:



X	Y1	Y2
12	560	360
13	540	390
14	520	420
15	500	450
16	480	480
17	460	510
18	440	540

X=16

   We could also solve $800 - 20x = 30x$ to get $x = 16$. Sixteen minutes after the fourth balloon launches, both balloons will be at the same height, 480 feet.

Extension Questions

- If the function for the motion of a fifth balloon were $y = 700 - 20x$, how would the movement of the fifth balloon have been different from the movement of the first balloon?

  The fifth balloon would be sighted at a height of 700 feet instead of 800 feet. The rate of descent would be the same as the rate of descent of the first balloon.

- Would the fifth balloon have landed sooner or later than the first balloon? Explain how you know.

  If the fifth balloon were sighted at a lower altitude than the first balloon but descended at the same rate, it would land sooner. The x-intercept would be 700 divided by 20, which is 35 seconds.

The Walk

Two motion detectors have been set up in a room so that two students, Pam and Abigail, may walk in parallel paths, each directly in front of a motion detector. The table below shows the data that was collected for each walk. Assume the students each walked at a constant rate and started at the same time.

Pam's Walk

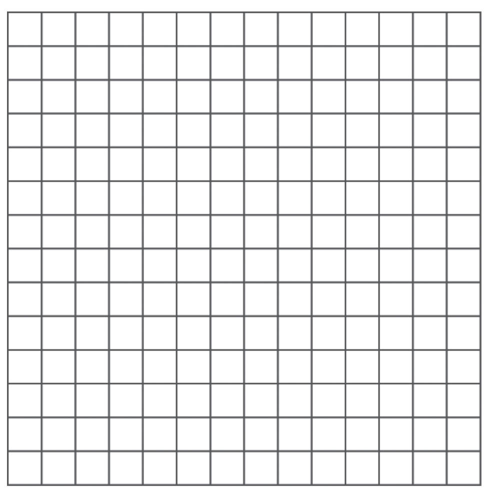
Time (seconds)	Distance (feet) from Motion Detector
1	7.9
3	5.3
6	1.4

Abigail's Walk

Time (seconds)	Distance (feet) from Motion Detector
2	3.6
4	5.2
7	7.6

1. What are reasonable domain and range values for this problem situation?

2. Create a graph to model the students' walks. Label the axes.



3. Write a function rule that models each student's distance from the motion detector in terms of the number of seconds.

4. Determine the point of intersection to the nearest tenth of the two lines.

5. Describe what the point of intersection represents in the scenario.

Notes

CCSS Content Task

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a

Scaffolding Questions

- How would you describe the shape of the graph if the students walked at a constant rate?
- How can you use the tables to determine how fast each student was walking?
- If you plot the points, what pattern do you see?
- How is Pam's walk different from Abigail's walk?

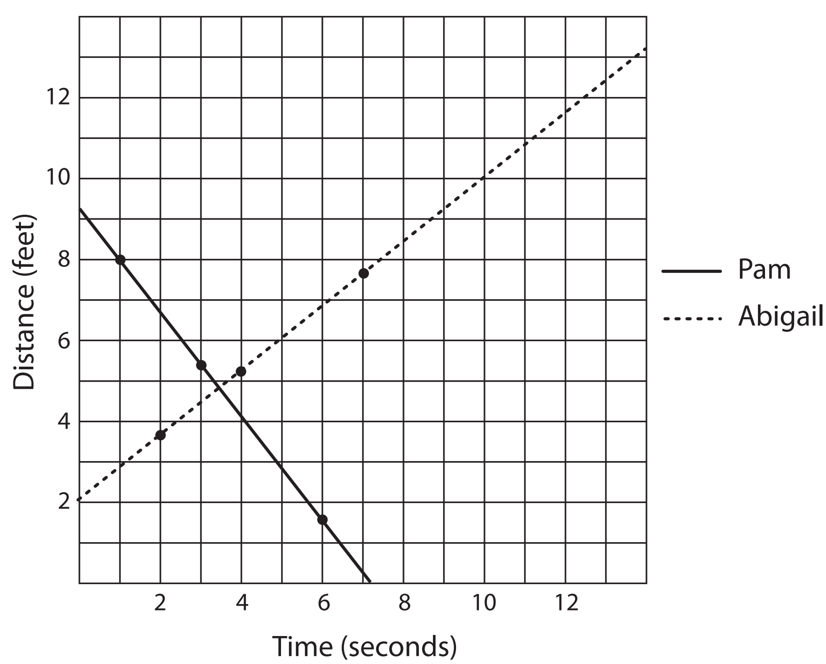
Sample Solutions

1. What are reasonable domain and range values for this problem situation?

   The domain values must be greater than or equal to 0. (In a real-life situation, there is a limit to the amount of time the students could walk in front of the motion detectors; they are limited by the room's size or the detectors' sensing range.)

   The range values represent distance and must be positive numbers. The distance or range is limited by detector's sensing range.

2. Create a graph to model the students' walks. Label the axes.



3. Write a function rule that models each student's distance from the motion detector in terms of the number of seconds.

   Start by using the differences between each student's recorded distances to determine how fast each one walked.

Pam's Walk

	Time (seconds)	Distance (ft) from Motion Detector	
	1	7.9	
2			-2.6
	3	5.3	
3			-3.9
	6	1.4	

Abigail's Walk

	Time (seconds)	Distance (ft) from Motion Detector	
	2	3.6	
2			1.6
	4	5.2	
3			2.4
	7	7.6	

   Pam walked at a rate of 1.3 feet per second, which is represented by –1.3. Pam's distance from the motion detector decreased, so she must have walked toward the detector.

   Abigail walked at a rate of 0.8 feet per second, represented by 0.8. Abigail's distance from the motion detector increased, so she must have walked away from the detector.

   The function rule for Pam's walk is the starting point plus the product of her rate and the number of seconds she walked. Use the table and the rates to determine the starting points. Add an extra row for time 0.

Pam's Walk

	Time (seconds)	Distance (ft) from Motion Detector	
	0	9.2	
1			-1.3
	1	7.9	
2			-2.6
	3	5.3	
3			-3.9
	6	1.4	

Abigail's Walk

	Time (seconds)	Distance (ft) from Motion Detector	
	0	2	
2			1.6
	2	3.6	
2			1.6
	4	5.2	
3			2.4
	7	7.6	

   To determine Pam's starting point, add 1.3 feet—the distance she traveled in 1 second—to 7.9 feet—her distance from the motion detector at 1 second. This shows where Pam was at 0 seconds.

   $1.3 + 7.9 = 9.2$

   At 0 seconds, Pam was 9.2 feet from the motion detector. The function rule for Pam's walk is $y = 9.2 – 1.3x$.

linear function in terms of the situation it models, and in terms of its graph or a table of values.

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(A-REI) **Solve systems of equations**

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(F-IF) **Interpret functions that arise in applications in terms of the context**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n)*

Notes

*gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

CCSS Additional Teacher Content

(8.F) **Define, evaluate, and compare functions.**

3. Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function A = s2 giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.

(A-SSE) **Interpret the structure of expressions**

1. Interpret expressions that represent a quantity in terms of its context.*

a. Interpret parts of an expression, such as terms, factors, and coefficients.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.*

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

(F-LE) **Interpret expressions for functions in terms of the situation they model**

5. Interpret the parameters in a linear or exponential function in terms of a context.*

Standards for Mathematical Practice

2. Reason abstractly and quantitatively.

4. Model with mathematics.

The function rule for Abigail's walk is her starting point plus the product of her rate and the number of seconds she walked. Her starting point is 3.6 minus the distance she traveled in 2 minutes.

$3.6 - 1.6 = 2$

At 0 seconds, Abigail was 2 feet from the motion detector. The function rule that describes Abigail's walk is $y = 2 + 0.8x$.

4. Determine the point of intersection to the nearest tenth of the two lines.

We want to know when Pam and Abigail were the same distance from their own motion detector. The graphs show that at about 3 seconds they were both about 5 feet from their own detector.

To check this solution, solve the system:

$y = 9.2 - 1.3x$

$y = 2 + 0.8x$

$$\begin{aligned} 2 + 0.8x &= 9.2 - 1.3x \\ 2.1x &= 7.2 \\ x &\approx 3.4 \end{aligned}$$

$y = 2 + 0.8(3.4) = 4.7$

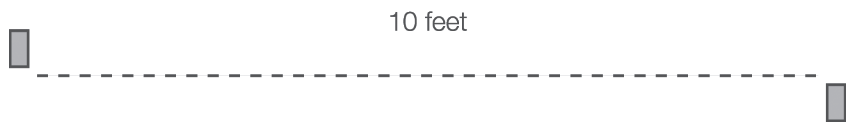
The point of intersection is (3.4, 4.7).

5. Describe what the point of intersection represents in the scenario.

The point of intersection represents when Pam and Abigail were both the same distance from their own motion detector. This happened at about 3.4 seconds, when they were both about 4.7 feet from their own motion detector.

Extension Questions

- Suppose the motion detectors were set up on opposite sides of the room 10 feet apart and the girls walked on parallel paths. If the same data were used, how would your answer to question 4 be different?



If the motion detectors were 10 feet apart on opposite sides of the room, the two walkers would walk in the same direction.

The equations describing their motion relative to each person's motion detector would still be

Pam: $y = 9.2 - 1.3x$

Abigail: $y = 2 + 0.8x$

However, to determine when they would be in the same horizontal position, one must write the equations in terms of distance from one of the motion detectors. Suppose the equations are written as distance from Abigail's motion detector with respect to time.

Abigail started 2 feet from her motion detector. Pam started 9.2 feet from her motion detector. Since the distance between the motion detectors is 10 feet, Pam would be on a horizontal distance of 10 – 9.2, or 0.8 feet from Abigail's motion detector. The equation of Pam's movement relative to Abigail's motion detector is her starting point plus the product of her rate and the number of seconds.

$y = 0.8 + 1.3x$

The rate is positive because Pam's distance from Abigail's motion detector is increasing.

Abigail's rule is $y = 2 + 0.8x$.

Solving this system of equations results in a solution of

$$\begin{aligned} 0.8 + 1.3x &= 2 + 0.8x \\ 0.5x &= 1.2 \\ x &= 2.4 \end{aligned}$$

$y = 2 + 0.8(2.4)$

$y \approx 3.92$

They would both be about 3.92 feet from Abigail's motion detector 2.4 seconds after they started walking.

- Consider the previous scenario. How would the function rules change if the motion detectors were positioned 12 feet apart instead of 10 feet?

  Pam would be 12 – 9.2, or 2.8 feet from Abigail's motion detector. The equation of her movement relative to Abigail's motion detector is her starting point plus the product of her rate and the number of seconds she walked.

  $y = 2.8 + 1.3x$

  Pam's equation would be $y = 2.8 + 1.3x$*.*

  Abigail's rule does not change. It is still $y = 2 + 0.8x$*.*

Chemistry Dilemma

Bonnie and Carmen are lab partners in a chemistry class. Their chemistry experiment calls for a 5-ounce mixture that is 65% acid and 35% distilled water. There is no pure acid in the chemistry lab, but the partners find two mixtures that are labeled as containing some acid. Mixture A contains 70% acid and 30% distilled water. Mixture B contains 20% acid and 80% distilled water.

How many ounces of each mixture should Bonnie and Carmen use to make a 5-ounce mixture that is 65% acid and 35% distilled water? Justify your solution using symbols, tables, and graphs.

Notes

CCSS Content Task

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a

Scaffolding Questions

- How much distilled water is there in 1 ounce of Mixture A? How do you know?
- How many ounces of acid are there in 4 ounces of Mixture A?
- How many ounces of distilled water are there in 2 ounces of Mixture B?
- What are the variables in this situation?

Sample Solutions

How many ounces of each mixture should Bonnie and Carmen use to make a 5-ounce mixture that is 65% acid and 35% distilled water? Justify your solution using symbols, tables, and graphs.

Table: (shown on next page)

This is a possible student solution showing combinations of Mixture A and Mixture B and the percentages of acid and water in the new mixture.

Amount of Mixture A	Amount of Mixture B	Amount of Acid in New Mixture	Amount of Distilled Water in New Mixture	Percent of New Mixture That Is Acid	Percent of New Mixture That Is Distilled Water	Is It 65% Acid and 35% Water?
1	5 – 1 = 4	0.7(1) + 0.2(4) = 1.5	0.3(1) + 0.8(4) = 3.5	1.5 out of 5 = 30%	3.5 out of 5 = 70%	Too much water
2	5 – 2 = 3	0.7(2) + 0.2(3) = 2	0.3(2) + 0.8(3) = 3	2 out of 5 = 40%	3 out of 5 = 60%	Too much water
3	5 – 3 = 2	0.7(3) + 0.2(2) = 2.5	0.3(3) + 0.8(2) = 2.5	2.5 out of 5 = 50%	2.5 out of 5 = 50%	Too much water
4	5 – 4 = 1	0.7(4) + 0.2(1) = 3	0.3(4) + 0.8(1) = 2	3 out of 5 = 60%	2 out of 5 = 40%	Too much water
5	0	0.7(5) + 0 = 3.5	0.3(5) + 0 = 1.5	3.5 out of 5 = 70%	1.5 out of 5 = 30%	Not enough water

linear function in terms of the situation it models, and in terms of its graph or a table of values.

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(A-REI) **Solve systems of equations**

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit

expression, a recursive process, or steps for calculation from a context.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.

4. Model with mathematics.

The amount of Mixture A that Bonnie and Carmen should use must be between 4 and 5 ounces.

4.5	0.5	0.7(4.5) + 0.2(0.5) = 3.25	0.3(4.5) + 0.8(0.5) = 1.75	3.25 out of 5 = 65%	1.5 out of 5 = 35%	The correct amounts are 4.5 ounces of Mixture A and 0.5 ounces of Mixture B.

Symbols:

Another approach to the problem, using symbols, requires that the variables be defined.

The variables are the amounts of Mixture A and Mixture B.

x = the number of ounces of Mixture A

y = the number of ounces of Mixture B

The total amount of the new mixture must be 5 ounces.

Equation 1: $x + y = 5$

The amount of acid in Mixture A plus the amount of acid in the Mixture B must be 65% of 5 ounces. The acid in Mixture A can be expressed as $0.7x$ and the acid in Mixture B as $0.2y$.

Equation 2: $0.7x + 0.2y = 0.65(5)$

Similarly, the amount of distilled water in Mixture A plus the amount of distilled water in Mixture B must be 35% of 5 ounces. The distilled water in Mixture A can be expressed as $0.3x$, and the distilled water in Mixture B as $0.8y$.

Equation 3: $0.3x + 0.8y = 0.35(5)$

To solve symbolically, use two of the equations and the substitution method:

$x + y = 5$

$0.3x + 0.8y = 0.35(5)$

$y = 5 - x$

$$\begin{aligned}
0.3x + 0.8\,(5 - x) &= 0.35(5)\\
0.3x + 4 - 0.8x &= 1.75\\
-0.5x &= -2.25\\
x &= 4.5\\
y &= 5 - 4.5 = 0.5
\end{aligned}$$

Bonnie and Carmen should use 4.5 ounces of Mixture A and 0.5 ounces of Mixture B to make 5 ounces of a new mixture that is 65% acid and 35% water.

Graphs:

Graphs can also be used to solve the problem. To use a graphing calculator to graph or make a table, solve the equations for y:

$x + y = 5$

$y = 5 - x$

$0.7x + 0.2y = 0.65(5)$

Solve for y.

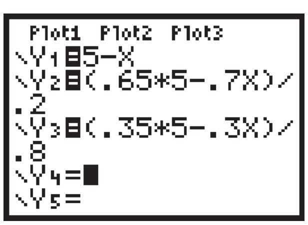
$$y = \frac{0.65(5) - 0.7x}{0.2}$$

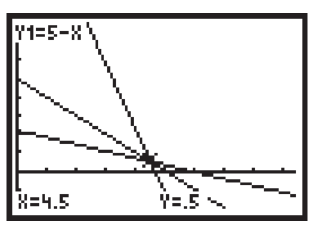
$0.3x + 0.8y = 0.35(5)$

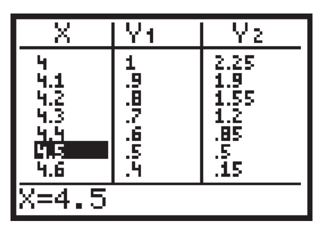
Solve for y.

$$y = \frac{0.35(5) - 0.3x}{0.8}$$

Graph the equations and make a table of values to find the common point, which is (4.5, 0.5). This means that Bonnie and Carmen should use 4.5 ounces of Mixture A and 0.5 ounces of Mixture B to make 5 ounces of a new mixture that is 65% acid and 35% water.







Extension Questions

- Was it necessary to have three equations to solve the problem?

  The second and third equations are complements of each other, since one gives the amount of acid needed and the other gives the amount of water needed. For instance, 70% water in one mixture means that the mixture is 30% acid. Since the amounts of water and acid must add up to 100%, an equation for both amounts is not necessary.

- Does it matter which pair of equations is used?

  The graph shows that it does not matter which pair of equations is used. If you use any pair of equations, their graphs intersect at the point (4.5, 0.5).

- What could have been determined if the total amount or 5 ounces was not given?

  The total amount could be expressed as $x + y$.

  The number 5 would be replaced by $x + y$ in the second and third equations.

  The amount of acid would be expressed as $0.7x + 0.2y = 0.65(x + y)$.

  Similarly, the amount of distilled water in Mixture A plus the amount of distilled water in Mixture B must be 65% of $x + y$ ounces.

  $$0.3x + 0.8y = 0.35(x + y)$$

  Simplify the equations:

  $$\begin{aligned} 0.7x + 0.2y &= 0.65x + 0.65y \\ 0.05x - 0.45y &= 0 \end{aligned}$$

  $$\begin{aligned} 0.3x + 0.8y &= 0.35x + 0.35y \\ 0.05x - 0.45y &= 0 \end{aligned}$$

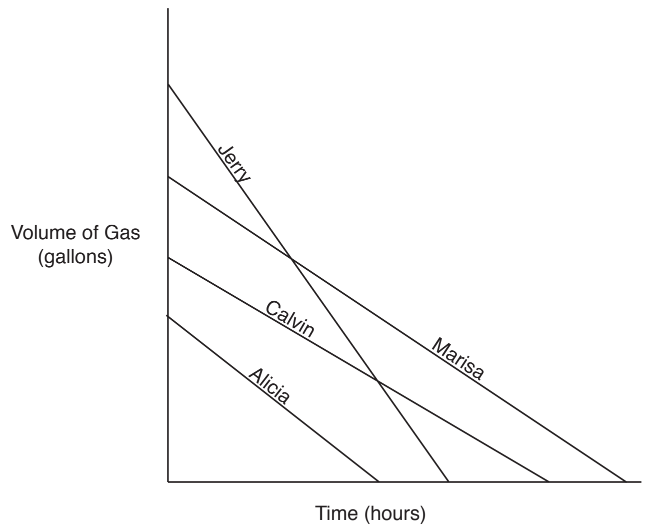
  The system could not be solved for specific values of x and y, but you would know what the ratio of x to y must be for any solution.

  $$\begin{aligned} 0.05x - 0.45y &= 0 \\ 0.05x &= 0.45y \\ \frac{x}{y} &= \frac{0.45}{0.05} = \frac{9}{1} \end{aligned}$$

  To meet the conditions of the problem, the amount of Mixture A must be 9 times the amount of Mixture B.

Four Cars

Jerry, Alicia, Calvin, and Marisa wanted to test their cars' gas mileage. Each person filled his or her car's gas tank to the maximum capacity and drove on a test track at 65 miles per hour until the car ran out of gas. The graphs given below show how the amount of gas in the cars changed over time.



1. Whose car has the largest gas tank? Explain your reasoning.
2. Whose car ran out of gas first? Explain your reasoning.
3. Whose car traveled the greatest distance? How does knowing that they all traveled at 65 miles per hour help you know who traveled the greatest distance?
4. Determine whose car gets the worst gas mileage. Describe how you used the graphs to make your decision.
5. How are Calvin's graph and Marisa's graph similar and how are they different in terms of the given situation?
6. In terms of the given situation, how does the function rule that describes Marisa's graph compare to the function rule that describes Jerry's graph?
7. In terms of the given situation, how does the function rule that describes Jerry's graph compare to the function rule that describes Alicia's graph?
8. Jerry's graph intersects Marisa's graph. What does the point of intersection represent?
9. After some automotive work, Jerry is now getting better gas mileage. How will this affect his graph?

Notes

CCSS Content Task

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

   a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

(8.F) **Define, evaluate, and compare functions.**

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

5. Describe qualitatively the functional relationship between two quantities by analyzing a

Scaffolding Questions

- Which car do you think uses the most gas?
- Which gas tank holds the most gas?
- Which gas tank holds the least gas?
- Do any of the cars get similar miles per gallon?
- Why are all the graphs in quadrant 1?
- Do these situations have positive or negative slopes?
- Which lines appear to be parallel?
- If two lines are parallel, what is the same in their equations?
- What does the slope of these graphs represent?
- What do the equations of these lines look like?
- Which person's graph has the greatest y-intercept?
- What does the y-intercept represent in this situation?
- Which person's graph has the greatest x-intercept?
- What does the x-intercept represent in this situation?

Sample Solutions

1. Whose car has the largest gas tank? Explain your reasoning.

   The given information states that each person filled his or her tank to capacity. Jerry's car has the largest tank because at 0 hours his car's gas tank had the greatest volume.

2. Whose car ran out of gas first? Explain your reasoning.

   Alicia's car ran out of gas first because it reached a volume of 0 in the shortest amount of time.

3. Whose car traveled the greatest distance? How does knowing that they all traveled at 65 miles per hour help you know who traveled the greatest distance?

   Marisa's car traveled the farthest because it took the longest time to reach a volume of 0. Each person was traveling at 65 miles per hour. Since distance traveled is rate multiplied by time, and Marisa's time was the greatest, she traveled the greatest distance.

4. Determine whose car gets the worst gas mileage. Describe how you used the graphs to make your decision.

   Jerry's car gets the worst gas mileage. We can tell because his graph has the steepest slope. This means his car's rate of change decreased at the fastest rate. The absolute value of his rate of change is the greatest. The rate of change represents the number of gallons used per hour of travel.

5. How are Calvin's graph and Marisa's graph similar and how are they different in terms of the given situation?

   It appears that the linear graphs for Calvin's and Marisa's cars' gas mileage are almost parallel, so the slopes are about the same. That is, the ratio of gallons per hour is approximately the same for both cars. Since the y-intercept of Calvin's graph is smaller than the y-intercept of Marisa's graph, Calvin's car has a smaller gas tank. Since Calvin's graph indicates a volume of 0 gallons in less time than Marisa's graph, Calvin's car ran out of gas sooner.

6. In terms of the given situation, how does the function rule that describes Marisa's graph compare to the function rule that describes Jerry's graph?

   The function rules for Marisa's and Jerry's cars are very different. Jerry's gas tank is larger than Marisa's, so the function rule (in the form $y = mx + b$) that represents his car's situation has a greater y-intercept. Because his car's graph has a steeper declining slope, his car uses gas at a faster rate; the absolute value of his car's rate of change is greater.

7. In terms of the given situation, how does the function rule that describes Jerry's graph compare to the function rule that describes Alicia's graph?

   Jerry's gas tank holds more than Alicia's gas tank, so the function rule (in the form $y = mx + b$) that represents Jerry's car's situation has a greater y-intercept, or b value. Jerry's car uses gas at a faster rate than Alicia's car; therefore, the slope, or m in the function rule, is negative, with a larger absolute value than the slope in the function rule for Alicia's car.

graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

(F-IF) **Interpret functions that arise in applications in terms of the context**

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

(F-IF) **Analyze functions using different representations**

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Notes

CCSS Additional Teacher Content

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

(F-IF.5) **Interpret functions that arise in applications in terms of the context**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.

3. Construct viable arguments and critique the reasoning of others.

8. Jerry's graph intersects Marisa's graph. What does the point of intersection represent?

   The point of intersection represents the time when Jerry's gas tank and Marisa's gas tank held the same amount of gas.

9. After some automotive work, Jerry is now getting better gas mileage. How will this affect his graph?

   The x-intercept on Jerry's graph would be a larger number. The slope of his graph would show a more gradual decline.

Extension Questions

- If a fifth line were added to the graph parallel to, but different from, the line for Alicia's car, what would you know about this fifth car?

  It uses gasoline at the same rate as Alicia's car, but it has a different tank capacity.

- Suppose everyone traveled at 55 miles per hour instead of 65 miles per hour. How would this affect the graphs?

  If they traveled at a slower rate, the amount of gas used per hour would decrease. The graphs would decline at a less steep slope and the x-intercepts would be greater.

- Suppose Calvin's car's graph could be represented by the rule $y = 30 - 6x$. What information do you now know about Calvin's car? When did his car run of gas? What would be a reasonable rule for Alicia's travel?

  The capacity of Calvin's car's tank is 30 gallons because the y-intercept of $y = 30 - 6x$ is 30. His car is using gasoline at a rate of 6 gallons per hour. At $0 = 30 - 6x$, x is 5, so the x-intercept is 5. This means that it takes him 5 hours to run out of gas. If he is traveling at 65 miles per hour, he will have traveled 65 • 5, or 325 miles.

The rules for the other drivers can be estimated using the intercepts.

The y-intercept for the graph of Alicia's car is about half of that for Jerry's car, or approximately 15 gallons.

If Jerry's car's x-intercept is 5, the x-intercept for the graph for Alicia's car is about 4.

$y = b + mx$
$y = 15 + mx$
$0 = 15 + 4m$
$m = -3.75$
$y = 15 - 3.75x$ *is a possible rule for Alicia's travel*

This function rule indicates that she is using gasoline at a rate of approximately 3.75 gallons per hour.

Graph It

1. Create a graph and write a possible function rule for each line described below. Use one set of axes to graph all three lines.

   Line A: The line has slope $-\frac{1}{2}$ and a y-intercept of 3.

   Line B: Any line that is parallel to Line A.

   Line C: The line has a y-intercept of 5 but a more steeply decreasing slope than Line A.

2. Name two points that lie on Line C.

3. What are the similarities and differences between the graphs of the lines?

4. Must any of the lines intersect? Justify your reasoning.

Notes

CCSS Content Task

(8.F) **Define, evaluate, and compare functions.**

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(A-REI) **Represent and solve equations and inequalities graphically**

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Scaffolding Questions

- What can be determined if you know that the y-intercept is 3?
- How does knowing the slope help you create a graph?
- What part of the function rule must stay constant to produce a parallel line?

Sample Solutions

1. Create a graph, and write a possible function rule for each line described below. Use one set of axes to graph all three lines.

   Line A: The line has slope $-\frac{1}{2}$ and a y-intercept of 3.

   Line B: Any line that is parallel to Line A.

   Line C: The line has a y-intercept of 5 but a more steeply decreasing slope than Line A.

Line A:

To draw the graph, mark the y-intercept of 3 and use the slope to determine another point on the graph. The slope $-\frac{1}{2}$ means that for every change in y of –1 unit, there is a change in x of 2 units. Another point is (2, 2).

The function rule of a line can be written in the form $y = mx + b$, where m is the slope and b is the y-intercept.

For this line, the rule is $y = -\frac{1}{2}x + 3$.

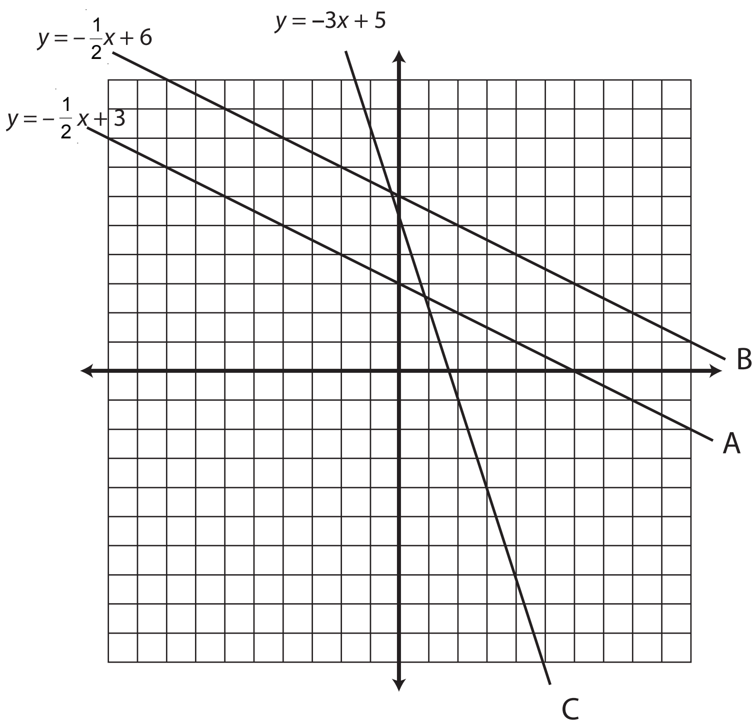
Line B:

If two lines are parallel, they have the same slope but a different y-intercept. The function rule for one possible parallel line is $y = -\frac{1}{2}x + 6$. Any line of the form $y = -\frac{1}{2}x + b$ for any real number b is correct.

Line C:

The *y*-intercept is 5, but the slope must be different. Increase the absolute value of the slope to get a steeper line. The line is still decreasing, so the slope must be negative. For example, if the slope is –3, the function rule is $y = -3x + 5$.

For the sample function rules, the graphs are as shown below. Note: Graphs may differ from those below for different function rules.



2. Name two points that lie on Line C.

   (1, 2) and (4, –7)

   Answers will vary depending on the functions students generate to satisfy the description of line C. For example, using the function rule $y = -3x + 5$, two points are (1,2) and (4, -7).

3. What are the similarities and differences between the graphs of the lines?

   The lines all have negative slopes and positive *y*-intercepts. Lines A and B have the same slope and

(F-IF) **Analyze functions using different representations**

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Standards for Mathematical Practice

7. Look for and make use of structure.

different *y*-intercepts. They are parallel lines. Line C intersects the other two lines; it has a slope of –3 (for example), and the *y*-intercept is 5.

4. Must any of the lines intersect? Justify your reasoning.

   Two lines always intersect if they have different slopes. Lines A and B, which have the same slope (and, therefore, are parallel), never intersect. Line C, which has a steeper declining slope, intersects Lines A and B.

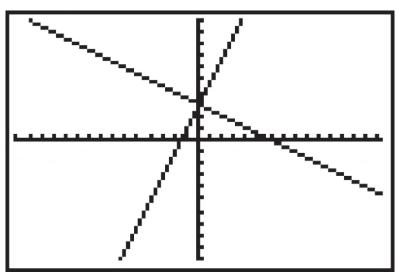
Extension Questions

- If another line is parallel to Line A and translated 6 units down, what is the function rule for the new line?

  The new line has the same slope, but its y-intercept changes to 3 – 6, or –3. The rule is $y = -\frac{1}{2}x - 3$.

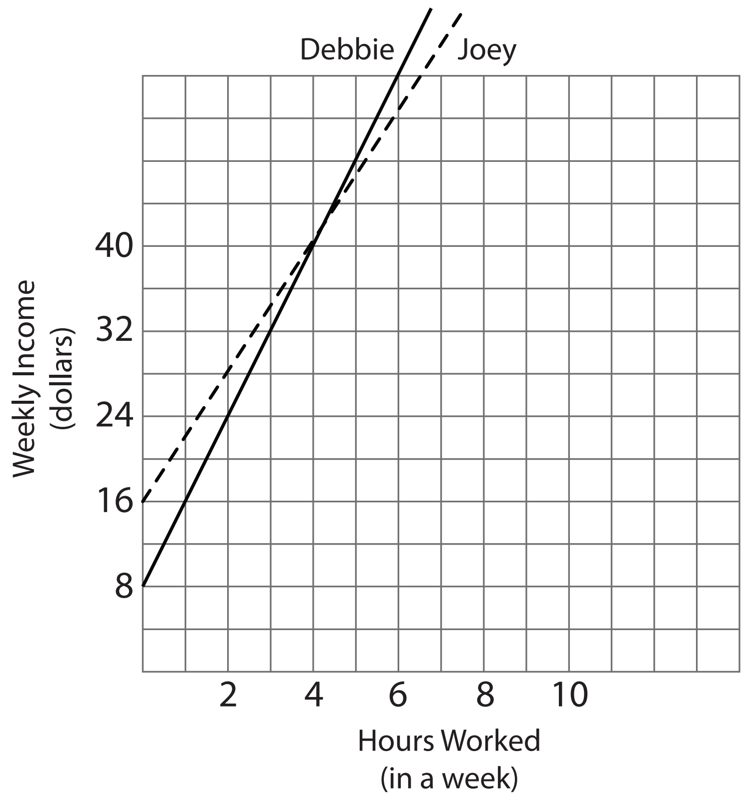
- What is the function rule of a line perpendicular to Line A with the same *y*-intercept?

  Perpendicular lines have opposite, reciprocal slopes. The slope of the new line is +2. The function rule of the new line is $y = 2x + 3$. *Students could check this answer with a graph.*



Summer Money

Debbie and Joey have decided to earn money during the summer. Each receives a weekly allowance and has also taken a job. The graphs model their weekly incomes, including allowance, as a function of the number of hours they work.



1. What information would you use to write a function rule?
2. Write a function rule that can be used to calculate the amount of money each person will earn per week in terms of the number of hours worked. Make a table of your data.
3. How will an increase in Debbie's allowance affect the graph? Entries in a table? The function rule? Use an example to justify your thinking.
4. How will an increase in Joey's hourly wages affect the graph? Entries in a table? The function rule? Use an example to justify your thinking.
5. If Debbie's weekly allowance is doubled, will her new weekly income be more or less than twice the original amount? Explain your reasoning.
6. Based on the original functions, who will have more money each week?

Notes

CCSS Content Task

(8.EE) **Understand the connections between proportional relationships, lines, and linear equations.**

6. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

(8.F) **Define, evaluate, and compare functions.**

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Scaffolding Questions

- How is each person's allowance represented on the graph?
- What are you using to represent the independent and dependent variables?
- What do you need to know to determine the function rule of a line?
- How much allowance does Debbie receive? How much allowance does Joey receive?
- What does the y-intercept represent for Debbie's situation? For Joey's situation?
- What does the y-intercept mean in the context of Debbie's and Joey's incomes?
- How do you find the slope from a graph? From a table?
- What is Debbie's salary per hour? Joey's salary per hour?
- What does the slope represent for each line?
- Describe in words how much money Debbie will earn per week.

Sample Solutions

1. What information would you use to write a function rule?

   Each person has a starting amount that will be the y-intercept of the function rule, or the b in $y = mx + b$. This represents the amount of allowance that each person receives. The slope of the function rule, represented as m in $y = mx + b$, is the rate of change per hour, which in this case is the amount of money each person gets paid per hour.

2. Write a function rule that can be used to calculate the amount of money each person will earn per week in terms of the number of hours worked. Make a table of your data.

   We can tell from the graph that Debbie's starting amount (her weekly allowance) is $8, and the rate of change from point (0, 8) to point (1, 16) is 8. So her hourly salary is \$8. The function rule for this line is $y = 8x + 8$.

The graph indicates that Joey's starting amount is \$16, and the rate of change from point (0, 16) to point (2, 28) is 12 for 2 hours. So his salary is \$6 per hour. The function rule for this line is $y = 6x + 16$.

Debbie's Income

Hours Worked	Income in Dollars
0	8
1	16
2	24
3	32
4	40
5	48
6	56
7	64
8	72

Joey's Income

Hours Worked	Income in Dollars
0	16
1	22
2	28
3	34
4	40
5	46
6	52
7	58
8	64

3. How will an increase in Debbie's allowance affect the graph? Entries in a table? The function rule? Use an example to justify your thinking.

   If Debbie's allowance increases, the *y*-values in the original table each need to be increased by the amount of the allowance change to make the *y*-values of the new table. For example, if Debbie's allowance increases by \$4.00, the corresponding income increases by \$4, as shown in the second table.

(A-SSE) **Interpret the structure of expressions**

1. Interpret expressions that represent a quantity in terms of its context.*
   a. Interpret parts of an expression, such as terms, factors, and coefficients.

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(F-IF) **Analyze functions using different representations**

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and

exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

(F-LE) **Interpret expressions for functions in terms of the situation they model**

5. Interpret the parameters in a linear or exponential function in terms of a context.

CCSS Additional Teacher Content

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(A-REI) **Solve systems of equations**

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(F-IF) **Interpret functions that arise in applications in terms of the context**

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.

4. Model with mathematics.

Debbie's Income

Hours Worked	**Income in Dollars**
0	8
1	16
2	24
3	32
4	40
5	48
6	56
7	64
8	72

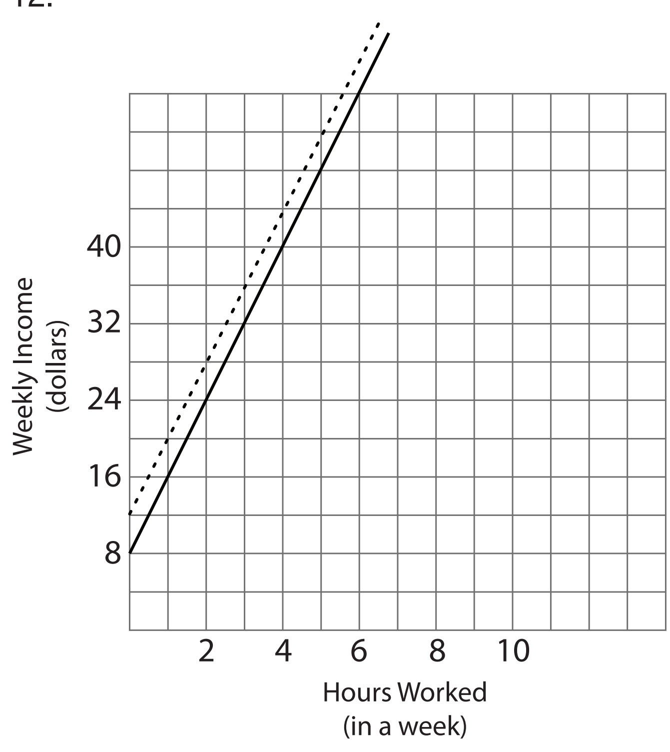
Debbie's Income with Allowance Increased by $4

Hours Worked	**Income in Dollars**
0	12
1	20
2	28
3	36
4	44
5	52
6	60
7	68
8	76

There is a difference of 4 in each y-value for the same x in the two tables. For example, the difference between the y-values for 8 hours in the two tables is 76 – 72, or 4.

In the function rule, the constant term changes. If Debbie's allowance increases by $4, the new allowance is $12, and the function rule is $y = 8x + 12$.

The graph of the new situation is a straight line parallel to the original line with a y-intercept of 12.



4. How will an increase in Joey's hourly wages affect the graph? Entries in a table? The function rule? Use an example to justify your thinking.

   If Joey's hourly rate increases by \$4, the hourly rate becomes \$10. His new function rule is $y = 10x + 16$.

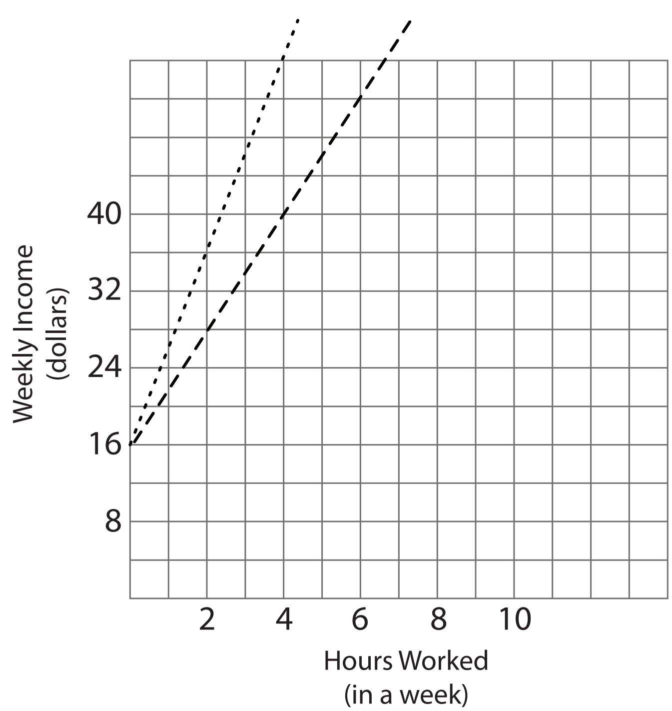
Joey's Income

Hours Worked	Income in Dollars
0	16
1	22
2	28
3	34
4	40
5	46
6	52
7	58
8	64

Joey's Income with Increase in Hourly Wage

Hours Worked	Income in Dollars
0	16
1	26
2	36
3	46
4	56
5	66
6	76
7	86
8	96

If Joey's hourly wages increase, the constant rate of change per hour worked increases. The slope of the linear graph gets steeper, and in the function rule the coefficient of x increases. This line has the same y-intercept since his allowance did not change, but it has a different slope.



5. If Debbie's weekly allowance is doubled, will her new weekly income be more or less than twice the original amount? Explain your reasoning.

   If Debbie's weekly allowance is doubled, her function rule changes from $y = 8x + 8$ to $y = 8x + (2)8$, or $y = 8x + 16$. Her hourly wage remains the same.

   The function rule for twice her original income is $y = 16x + 16$.

   For any positive number x, $8x + 16 < 16x + 16$.

   The new income from doubling Debbie's allowance is *less* than twice her original income because doubling her allowance does not affect her hourly wage, but doubling her original income increases both her hourly wage and her allowance.

6. Based on the original functions, who will have more money each week?

   It depends on how many hours they work. If they work more than 4 hours, Debbie will earn more money. The y-values (weekly income) on Debbie's line are greater when x is greater than 4. If they work fewer than 4 hours, Joey will earn more money.

Extension Questions

- What are reasonable domain values for this function and the problem situation?

  Domain values for the situation must be numbers greater than or equal to 0. If Debbie and Joey must work only whole hours, the domain values will be whole numbers. If they can work and get paid for portions of an hour, the domain of the situation could be all real numbers greater than or equal to 0.

- A line is used to model the situation. Will all points on this graph represent the problem situation?

  If a person is usually paid for whole numbers of hours or perhaps half hours worked, not all points on the line represent the situation. Rather, the graph of the problem situation is a step function that rises in increments, not the whole line.

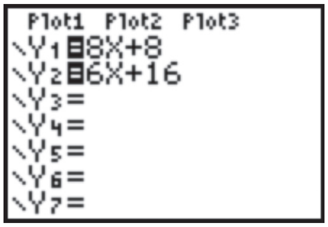
- Determine the point of intersection of the two lines. What does this point mean in the situation?

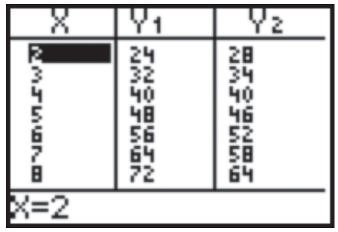
  The point of intersection is (4, 40), meaning that if Debbie and Joey both worked four hours in a week, they would have the same weekly income.

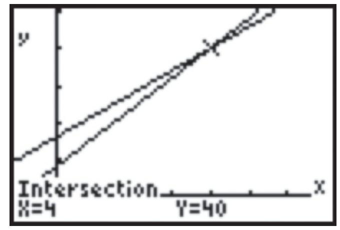
- How would you solve this problem using a graphing calculator?

  The point of intersection appears to be the point (4, 40). This can be verified by examining the table or graph on the calculator or by substituting into the given functions.

Symbolic | *Table* | *Graph*





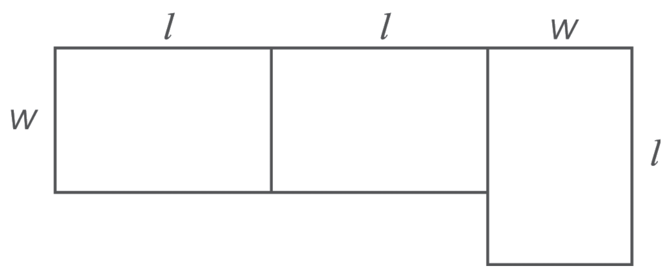


$$y = 6x + 16$$
$$40 = 6(4) + 16$$
$$y = 8x + 8$$
$$40 = 8(4) + 8$$

The point of intersection is (4, 40). This means that when Debbie and Joey work 4 hours, they both earn $40. Interpreting data from the graph, Joey earns more if they both work fewer than 4 hours; Debbie earns more if they both work more than 4 hours.

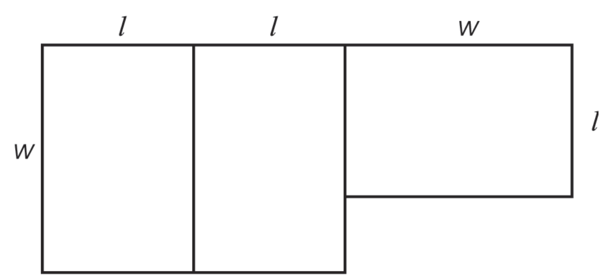
Exercise Pens

Devin is planning to build exercise pens for his three horses. Given the space available, he has decided to create three rectangular pens, as shown in the diagram. The three rectangular pens have the same size and shape. Devin has been advised that the perimeter of each pen should be 440 feet. He will use 1,200 feet of fencing for this project.



1. Use two different representations (tables, symbols, or graphs) to determine what the dimensions (in feet) of each pen should be.

2. Devin changes the orientation of the pens as shown below. What are the dimensions (in feet) of each pen? Describe how your answer relates to the problem situation.



Notes

CCSS Content Task

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(A-REI) **Solve systems of equations**

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

(A-REI) **Represent and solve equations and inequalities graphically**

Scaffolding Questions

- What are the known and unknown quantities in this situation?
- What does each known quantity represent?
- How do the perimeters of pens relate to the 1,200 feet of fencing?
- How many different relationships are described in the problem?

Sample Solutions

1. Use two different representations (tables, symbols, or graphs) to determine what the dimensions (in feet) of each pen should be.

The unknown dimensions are the length and width of the pen. In the diagram, the width of each pen is represented by w, and the length of each pen is represented by l.

The perimeter of each pen must be 440 feet. The perimeter is twice the length plus twice the width.

Equation 1:

$2l + 2w = 440$

Divide by 2:
$l + w = 220$

The amount of fencing must be equal to 1,200 feet. Fencing the pens requires six lengths and four widths.

Equation 2:

$6l + 4w = 1{,}200$

Divide by 2:
$3l + 2w = 600$

Subtract the first equation from the second equation to solve for length.

$$
\begin{array}{rcl}
3l + 2w & = & 600 \\
-(2l + 2w & = & 440) \\
\hline
l & = & 160
\end{array}
$$

Substitute that length value into the simplified version of Equation 1 to solve for *w*.

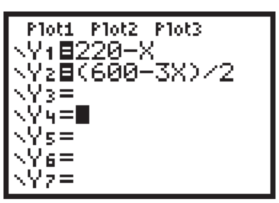
$$
\begin{array}{rcl}
l + w & = & 220 \\
160 + w & = & 220 \\
w & = & 60
\end{array}
$$

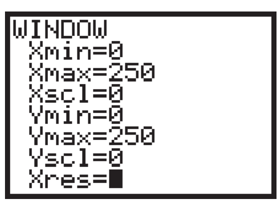
Each pen should have dimensions of 60 feet in width and 160 feet in length.

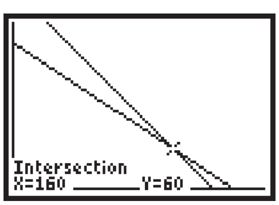
Another method for finding the pens' dimensions is to solve each equation for width, graph the functions, and find the point of intersection.

$$
\begin{array}{rcl}
l + w & = & 220 \\
w & = & 220 - l \\
3l + 2w & = & 600 \\
w & = & \frac{600 - 3l}{2}
\end{array}
$$

Enter the function rules into the graphing calculator. Let the width be the *y*-value and the length be the *x*-value. Draw the graphs on the graphing calculator and find the point of intersection.







The point of intersection is (160, 60). The *x*-value is the length of 160 feet, and the *y*-value is the width of 60 feet.

11. Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

CCSS Additional Teacher Content

(A-SSE) **Interpret the structure of expressions**

1. Interpret expressions that represent a quantity in terms of its context.*

a. Interpret parts of an expression, such as terms, factors, and coefficients.

(F-IF) **Interpret functions that arise in applications in terms of the context**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

(F-LE) **Interpret expressions for functions in terms of the situation they model**

5. Interpret the parameters in a linear or exponential function in terms of a context.*

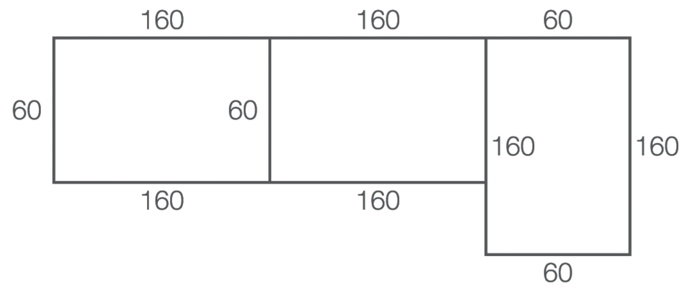
Notes

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.

7. Look for and make use of structure.

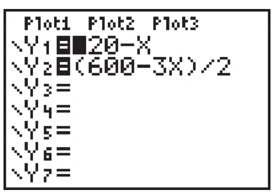
The fenced area would look like this diagram.

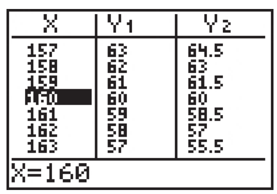


The perimeter of each pen is 2(60) + 2(160) = 440 feet.

The total amount of fencing required is 4(60) + 6(160) = 1,200 feet.

Some students may also use a table to determine the point of intersection. The equations can be entered into the graphing calculator, and then students look for the point at which the *y*-values for each function are equal, which occurs at the point (160, 60).





2. Devin changes the orientation of the pens as shown below. What are the dimensions (in feet) of each pen? Describe how your answer relates to the problem situation.

   The perimeter of each pen is still represented by the equation $2w + 2l = 440$, or $w + l = 220$.

   The amount of fencing for this figure is represented by $5w + 5l$. The total amount of fencing is 1,200 feet.

   $5w + 5l = 1200$, or $w + l = 240$

   The system of equations is

   $w + l = 220$

   $w + l = 240$

   This system has no solution. The initial restriction on the perimeter, $w + l = 220$, is contradicted by the second equation, $w + l = 240$. This means that both conditions cannot be met with one set of dimensions. In the configuration of $w + l = 240$, the perimeter of each pen will be 480 feet, which is larger than the initial restriction of 440 feet for each pen.

Extension Questions

- Two function rules were used to create lines to represent the situation. Describe the domains and ranges for the functions and the domains and ranges for the problem situation.

  The domain and range of each linear function is the set of all real numbers. However, for the problem situation, the domain and range values are restricted to first quadrant values.

  $y = 220 - x \quad 0 < x < 220 \quad 0 < y < 220$

  $y = \frac{600 - 3x}{2} \quad 0 < x < 200 \quad 0 < y < 300$

  For the two functions together, the domain is restricted to the intersection of the two domains $0 < x < 200$ and the range is restricted to $0 < y < 220$.

- How would your equations change for the situation in question 1 if the total amount of fencing were 800 feet?

  The first equation would not be different, but the equation for total amount of fencing would become $6l + 4w = 800$, or $3l + 2w = 400$.

- Solve the system from the previous extension question and explain the solution.

  One solution method is to subtract the first equation from the second equation.

  $$\begin{array}{rcl} 3l + 2w & = & 400 \\ 2l + 2w & = & 440 \\ \hline l & = & -40 \end{array}$$

  Since l represents the dimension of a rectangle, it cannot be negative. A total of 800 feet of fencing is not enough for each pen to have a perimeter of 440 feet.

- How would your equations change for the situation in question 1 if the total amount of fencing were 1,000 feet?

  The second equation becomes $6l + 4w = 1{,}000$, or $3l + 2w = 500$.

  One solution method is to subtract the first equation from the second equation.

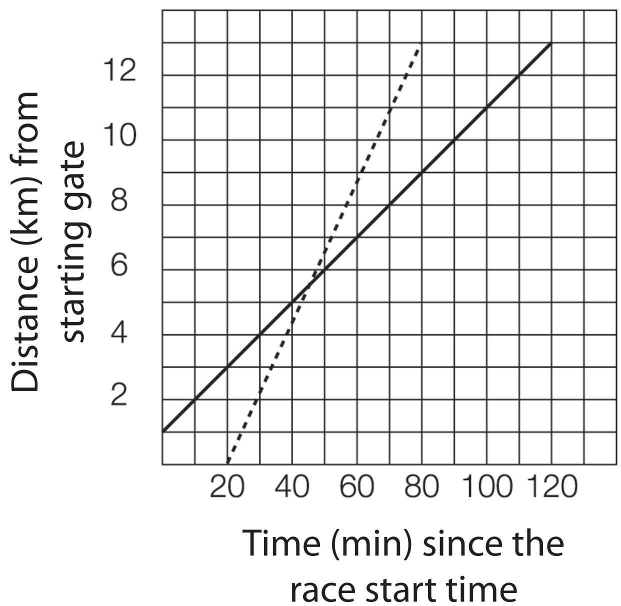
  $$\begin{array}{rcl} 3l + 2w & = & 500 \\ 2l + 2w & = & 440 \\ \hline l & = & 60 \end{array}$$

  Substitute that length value into the simplified version of the first equation to solve for w.

  $$\begin{array}{rcl} w & = & 220 - 60 \\ w & = & 160 \end{array}$$

The Run

The figure below represents the average speeds for two participants in a race—Eloise and Ty. The solid line segment represents Eloise's average speed, and the dashed line segment represents Ty's average speed. The graphs do not imply that each participant maintained a constant speed, since speed varies over time according to terrain, fatigue, or other factors.



1. Study the graphs and write a description of each participant's progress during the race.
2. Write a function rule that models each participant's progress.
3. For each participant, describe the meaning of the slope and intercepts represented in the graph.
4. What is the point of intersection of the two line segments? What does it represent?

Notes

CCSS Content Task

(8.EE) **Understand the connections between proportional relationships, lines, and linear equations.**

6. Use similar triangles to explain why the slope *m* is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at *b*.

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

   b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.*

   c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine

Scaffolding Questions

- How do you determine the rate at which Eloise is traveling?
- What is Ty's average speed?
- What is the length of the race? How do you know?
- Which participant took the least amount of time to complete the race?

Sample Solutions

1. Study the graphs and write a description of each participant's progress during the race.

   Possible description:

   Eloise started when the time was 0 and was positioned 1 kilometer from the starting gate. That first point on the line segment is (0, 1). Since her average speed is represented as a line segment, choose any two points on the line segment to determine the rate. For example, the points (0, 1) and (10, 2) can be used to find the rate. The rate of change from the point (0, 1) to the point (10, 2) is 1 kilometer in 10 minutes, or $\frac{1}{10}$ of a kilometer per minute.

   Ty left 20 minutes after Eloise. The first point on Ty's line segment is (20, 0). One reason for this could be that he arrived late at the race site or experienced some equipment problems such as difficulty with his shoes. Another point on that line is (80, 13). The rate of change from (20, 0) to (80, 13) is 13 kilometers in 60 minutes. The rate is about 4.61 minutes per kilometer, or $\frac{13}{60}$ kilometers per minute.

2. Write a function rule that models each participant's progress.

   The function rule for Eloise is the starting value (1) plus the product of the rate ($\frac{1}{10}$) and the number of minutes (x).

   $y = 1 + \frac{1}{10}x$

   Each line segment stops when *y* is 13; therefore, the race is 13 kilometers long. The *y*-value for Eloise is 13 when x is 120 minutes. Eloise started the race at the 1-kilometer mark, so she traveled 12 kilometers in 120 minutes.

Ty left the starting gate at 20 minutes after the race start time and traveled at $\frac{13}{60}$ kilometers per minute. Using the point-slope equation of a line, the function rule for Ty is

$y = \frac{13}{60}(x - 20) + 0$, or

$y = \frac{13}{60}x - 4\frac{1}{3}$

3. For each participant, describe the meaning of the slope and intercepts represented in the graph.

The slopes of the lines represent the average speed at which each person traveled.

Ty traveled 13 kilometers per 60 minutes or $\frac{13}{60}$ of a kilometer in 1 minute. Eloise traveled at 1 kilometer per 10 minutes or $\frac{1}{10}$ of a kilometer in 1 minute.

Eloise's y-intercept indicates that she was 1 kilometer from the starting gate at 0 minutes. Her x-intercept can be inferred from the graph to be (–10, 0), which does not make sense for this situation since time a time of –10 isn't possible. Ty's y-intercept can be inferred to be (0, –4), which also does not make sense since a distance of –4 is not possible. His x-intercept is (20, 0), which means that at 20 minutes after the race start time, Ty was at the starting gate.

4. What is the point of intersection of the two line segments? What does it represent?

The graphs of $y = \frac{1}{10}x + 1$ (Eloise) and $y = \frac{13}{60}(x - 20) + 0$ (Ty) intersect at about (45, 5.5).

To determine the exact values, solve the equation:

$\frac{1}{10}x + 1 = \frac{13}{60}(x - 20) + 0$

Multiply both sides of the equation by 60.

$6x + 60 = 13(x - 20)$

$6x + 60 = 13x - 260$

the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

(A-REI) **Solve systems of equations**

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find

the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where *f*(*x*) and/or *g*(*x*) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(F-IF) **Interpret functions that arise in applications in terms of the context**

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.*
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Standards for Mathematical Practice

2. Reason abstractly and quantitatively.

4. Model with mathematics.

$-7x = -320$

$x = \frac{-320}{-7}$

$x = 45\frac{5}{7}$

To find the *y*-value, substitute into the original equation for Eloise:

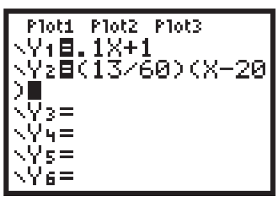
$y = \frac{1}{10}x + 1$

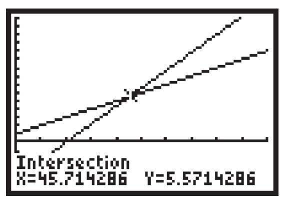
$y = \frac{1}{10}\left(\frac{320}{7}\right) + 1$

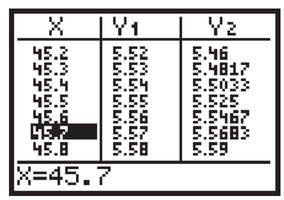
$y = 5\frac{4}{7}$

The point of intersection represents the point at which Eloise and Ty were the same distance from the starting line at the same time. The two participants met $5\frac{4}{7}$ kilometers from the starting line $45\frac{5}{7}$ minutes after the race start time. At this point, Ty overtook Eloise and stayed ahead for the rest of the race.

A graphing calculator may also be used to determine the point of intersection.



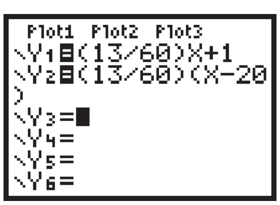


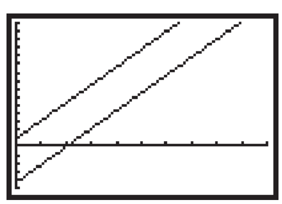


Extension Questions

- Describe ways in which the Eloise could have won the race.

  If she had traveled at the same rate as Ty, he would never catch up because she started first.





Any rate for Eloise that would determine a line that intersects with Ty's line after 13 kilometers would allow her to win. (This is a good calculator exploration of slope at the Algebra I level.)

Or students could reason that Eloise would have to run faster than $\frac{1}{10}$ of a *kilometer per minute and finish the race in less than 80 minutes. To find a possible rate, substitute the x and y values of (80, 13) in the function rule to find the slope:*

$13 = a(80) + 1$

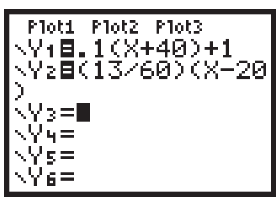
$a = \frac{12}{80}$

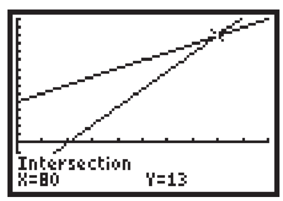
$a = \frac{3}{20}$

Eloise would have to run faster than $\frac{3}{20}$ of a kilometer per minute to win the race.

- If both participants traveled at the same rate, what factors or variables could be changed so that Eloise wins the race?

  The given graph shows that Eloise finished 40 minutes after Ty. If her graph is moved 40 units to the left, (x + 40) replaces x and both participants finish at the same time. That is, Eloise needs a head start of an additional 40 minutes to tie with Ty.





Thus, to win the race, Eloise's rule would have to be changed to reflect an amount greater than 40. For example, replace x with x + 41.

$y = \frac{1}{10}(x + 41) + 1$

$y = \frac{1}{10}x + 4.1 + 1$

$y = \frac{1}{10}x + 5.1$

She would need to start 5.1 kilometers from the starting line to win.

Which Plan Is Best?

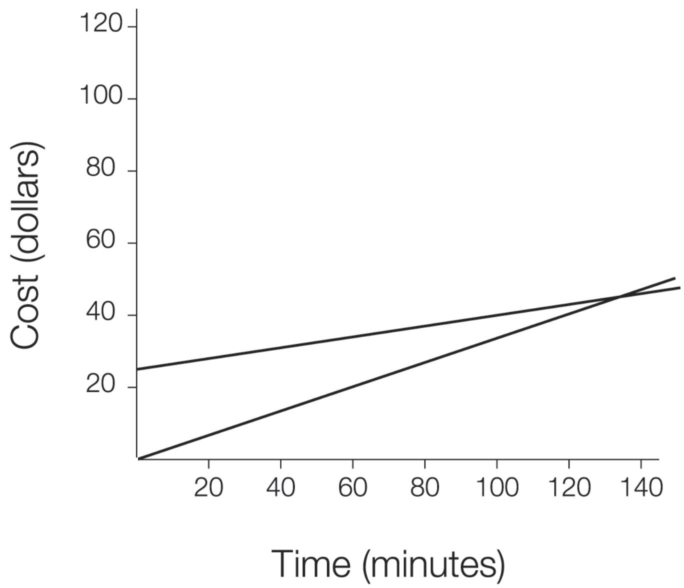
Students were given two cell phone plans to compare.

Plan 1: $C = \$0.35m$

Plan 2: $C = \$0.15m + \$25$

C represents the monthly cost in dollars, and m represents the time in minutes.

One group of students graphed the two plans.



1. Explain the differences between the two phone plans.
2. Explain the meaning of the slope for each plan.
3. Explain the meaning of the y-intercept for each plan.
4. Which plan offers the better deal? Explain your thinking.
5. If the second plan charged 20 cents per minute, what would be different about its graph?
6. If the first plan were changed so that the base fee was $10, how would its graph change?

Notes

CCSS Content Task

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

(8.F) **Use functions to model relationships between quantities.**

5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

(F-IF) **Interpret functions that arise in applications in terms of the context**

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

Scaffolding Questions

- How can you determine the slope for each cell phone plan?
- How is the cost per minute represented in each plan?
- Does either plan have a base or starting fee? How do you know?
- If you talk for 40 minutes, which plan costs more? How do you know?
- What does the point of intersection of the two lines mean in the context of this problem?

Sample Solutions

1. Explain the differences between the two phone plans.

   Plan 1 charges \$0.35 per minute, while Plan 2 charges only \$0.15 per minute. Plan 2 also has a base fee of \$25, and Plan 1 does not have a base fee.

2. Explain the meaning of the slope for each plan.

   In this situation, the slope of each plan's line represents the rate of change per minute, or the cost per minute in each plan. Plan 1's slope is 0.35. Plan 2's slope is 0.15.

3. Explain the meaning of the *y*-intercept for each plan.

   In this situation, the *y*-intercept represents each plan's cost at 0 minutes. Plan 1's *y*-intercept is 0 because there is no charge for the plan itself. Plan 2's *y*-intercept is 25 because there is a \$25 base fee connected with this plan.

4. Which plan offers the better deal? Explain your thinking.

   The better deal depends on the number of minutes you plan to use.

   Examine the table for the two plans:

X	Y_1	Y_2
123	43.05	43.45
124	43.4	43.6
125	43.75	43.75
126	44.1	43.9
127	44.45	44.05
128	44.8	44.2
129	45.15	44.35

X=123

If you plan to use fewer than 125 minutes, you should go with Plan 1. If you plan to use more than 125 minutes, the best plan would be Plan 2.

5. If the second plan charged 20 cents per minute, what would be different about its graph?

   The slope of the graph would change. The new graph would have a steeper slope.

6. If the first plan were changed so that the base fee was \$10, how would its graph change?

   The *y*-intercept of the original Plan 1 is 0. If a base fee of \$10 were added, the *y*-values for each point would increase by 10 units. The *y*-intercept would be 10.

Extension Questions

- What ways can Plan 1's method of charging change so that it is always a better deal than Plan 2?

  In general, if Plan 1 had a slope value less than 0.15 and a y-intercept less than or equal to 25, it would always be a better deal than Plan 2.

  Examples:

  If Plan 1 charged the same base rate as Plan 2 but decreased the slope, Plan 1's fee would always be less. For example, Plan 1 could charge 14 cents per minute with a base fee of \$24.99.

  $C = 0.14m + 24.99$

  Another option would be for Plan 1 to charge the same rate per minute as Plan 2 (15 cents) but decrease the base fee of \$25. Plan 1's price would be less for any number of minutes. For example, Plan 1 could charge 15 cents per minute with a base fee of \$20.

  $C = 0.15m + 20$

- Suppose the cell phone companies that are offering these plans merge. Together they come up with a new plan. The new plan has no base fee, charges the customer \$0.25 per minute, and provides the first 40 minutes free. The customer doesn't start paying until

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

CCSS Additional Teacher Content

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

Standards for Mathematical Practice

3. Construct viable arguments and critique the reasoning of others.

he or she has used the phone for more than 40 minutes. What is the function rule for this new plan?

First, make a table.

Minutes	*Cost*
40	*$0.00*
41	*$0.25*
42	*$0.50*
43	*$0.75*
44	*$1.00*

To figure out a function rule, we need to account for the 40 free minutes. The charge after 40 minutes is 25 cents per minute, or $2.50 for 10 minutes. Extend the table to determine the y-intercept (example below). There are negative costs for less than 40 minutes. These negative costs represent the money the customer is not paying or the money the customer is saving. Continuing to backtrack in the table, we learn that at 0 minutes, the cost is –$10.00.

Minutes	*Cost*
0	*–$10.00*
10	*–$7.50*
20	*–$5.00*
30	*–$2.50*
40	*$0.00*
41	*$0.25*
42	*$0.50*
43	*$0.75*
44	*$1.00*

The y-intercept is –10. Our function rule is $C = 0.25x - 10$, where x is the number of minutes and is greater than or equal to 40.

A Linear Programming Problem: Parking at the Mall

A new mall with 2 major department stores and 55 specialty shops is being built. You are a subcontractor in charge of planning and building the parking lots for the mall. The planners provide you with the following information:

- The total number of parking spaces must range from 2,000 to 2,400 spaces.
- For every employee parking space there must be at least 9 public parking spaces.
- There must be at least 20 employee parking spaces per department store and 2 employee parking spaces per specialty shop.

You anticipate that building costs will be $580 per public parking space and $600 per employee parking place.

The mall planners expect that, during an average week, revenue (average customer spending) from each public parking space will be at least $1,000 and from each employee parking space will be at least $100.

Design a proposal to present to the mall planners showing the feasible numbers of public and employee parking spaces. How many parking spaces of each type should be built to minimize the cost of building the parking lot? How many parking spaces of each type should be built to maximize weekly revenue?

Notes

Connections to the CCSS

(A-CED) **Create equations that describe numbers or relationships**

1. Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.**

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.**

CCSS Additional Teacher Material

(F-IF) **Interpret functions that arise in applications in terms of the context**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

Scaffolding Questions:

- What are the independent variables in this situation?
- Describe the restrictions (constraints) on the independent variables.
- How will you write these restrictions algebraically?
- What do these restrictions have to do with “the feasible region?”
- How will you go about graphing these restrictions?
- What does the cost of building the parking lot depend on? What function can you write for cost?
- What does the weekly revenue (average weekly customer spending per space) depend on? What function can you write for revenue?
- What is the Corner Principle for Linear Programming?
- What representations can you use to organize your proposal and answer the questions?

Sample Solutions:

Let *x* = the number of public parking spaces and *y* = the number of employee parking spaces.

To determine the feasible number of parking spaces to build, we need to describe the constraints on *x* and *y*.

Since the total number of parking spaces must be between 2,000 and 2,400,

$$2{,}000 \le x + y \le 2{,}400.$$

For every employee space there must be at least 9 public spaces, so

$$x \ge 9y \text{ or } y \le \frac{1}{9}x.$$

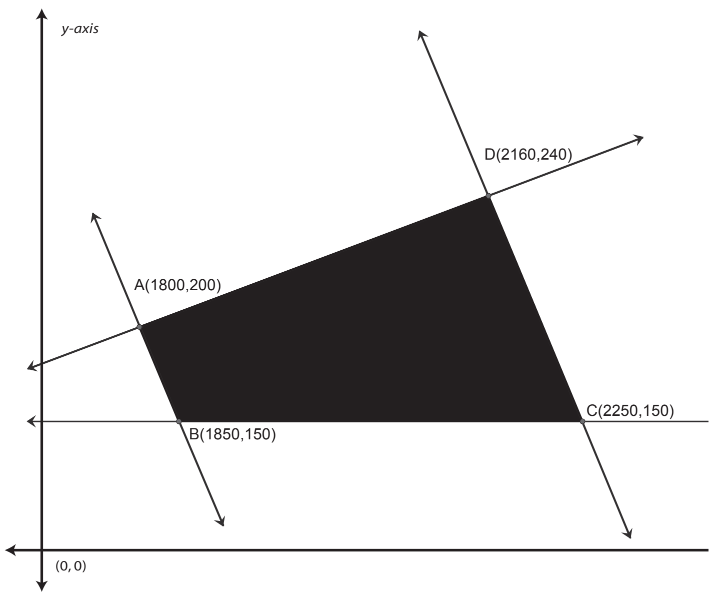
Finally, since we need at least 20 employee spaces for each of the 2 department stores and at least 2 employee spaces for each of the 55 specialty shops, we know that

$$y \geq 20(2) + 2(55)$$
$$y \geq 150.$$

The following restrictions are placed on the 2 variables:

$$y \geq 2,000 - x$$
$$y \leq 2,400 - x$$
$$y \leq \frac{1}{9}x$$
$$y \geq 150$$

The graph of the feasible region is shown below:



(Note: A good window for a calculator graph of this feasible region is $1,750 \leq x \leq 2,300, \quad 125 \leq y \leq 275$)

The points of intersection of the boundary lines are found by solving the systems.

Point A:

$$\begin{cases} x + y = 2,000 \\ x = 9y \end{cases} \Rightarrow \quad 9y + y = 2,000$$

$$10y = 2,000$$
$$y = 200$$
$$x = 9 \bullet 200$$
$$= 1,800$$

Standards for Mathematical Practice
3. Construct viable arguments and critique the reasoning of others.

6. Attend to precision.

Point B:

$$\begin{cases} x + y = 2{,}000 \\ y = 150 \end{cases} \Rightarrow x + 150 = 2{,}000$$

$$x = 1{,}850$$
$$y = 150$$

Point C:

$$\begin{cases} x + y = 2{,}400 \\ y = 150 \end{cases} \Rightarrow x + 150 = 2{,}400$$

$$x = 2{,}250$$
$$y = 150$$

Point D:

$$\begin{cases} x + y = 2{,}400 \\ x = 9y \end{cases} \Rightarrow 9y + y = 2{,}400$$

$$\begin{aligned} 10y &= 2{,}400 \\ y &= 240 \\ x &= 9 \bullet 240 \\ &= 2{,}160 \end{aligned}$$

All points in the feasible region and on the boundary of the region with integer coordinates would be feasible numbers of public and employee parking spaces.

To predict the minimum cost of building the parking lot, we need to write a cost function, *C*. Since it costs \$580 per public space and \$600 per employee space, the cost function is given by

$$C = C(x, y) = 580x + 600y.$$

The Corner Principle in Linear Programming tells us that the extreme (minimum and maximum) will occur at one of the vertices of the region. The following table gives *C* in thousands of dollars:

Vertex	*x*	*y*	*C(x,y)* = 580*x* + 600*y*
A	1,800	200	1,164
B	1,850	150	1,163
C	2,250	150	1,395
D	2,160	240	1,396.8

To minimize the cost of building the parking lot, there should be 1,850 public spaces and 150 employee spaces. However, the difference in the cost *s* for points A and B is only $1, so actually the selection of any point on the line segment from A to B would give a minimal cost. The cost will be about $1,163,000.

To maximize the weekly revenue, we need a weekly revenue function, *R*. Since we anticipate at least $1,000 per week per public space and $100 per week per employee space, that function is

$$R(x,y) = 1{,}000x + 100y$$

We apply the Corner Principle to this function, showing the weekly revenue in thousands of dollars:

Vertex	*x*	*y*	*R(x,y)* = 1000*x* + 100*y*
A	1,800	200	1,820
B	1,850	150	1,865
C	2,250	150	2,265
D	2,160	240	2,184

The weekly revenue will be maximized at $2,265,000 if the parking lot has 2,250 public spaces and 150 employee spaces.

The cost of making the parking lot may be the least at points A or B, but this is a one-time cost. However, the revenue is computed weekly. Thus, the proposal is to maximize the revenue by constructing 2,250 public spaces and 150 employee spaces.

Extension Questions:

- How can you investigate the cost of various combinations of public and employee parking spaces within the feasible region?

  You could make a table of the x- and y-coordinates of various points in the region and compute the corresponding costs.

- Would this be efficient to do?

  No. There are many points in the feasible region with integer coordinates.

- Instead of choosing points, what else might you choose?

  You could choose different values for the cost. Let C be the chosen cost and investigate the equation $580x + 600y = C$.

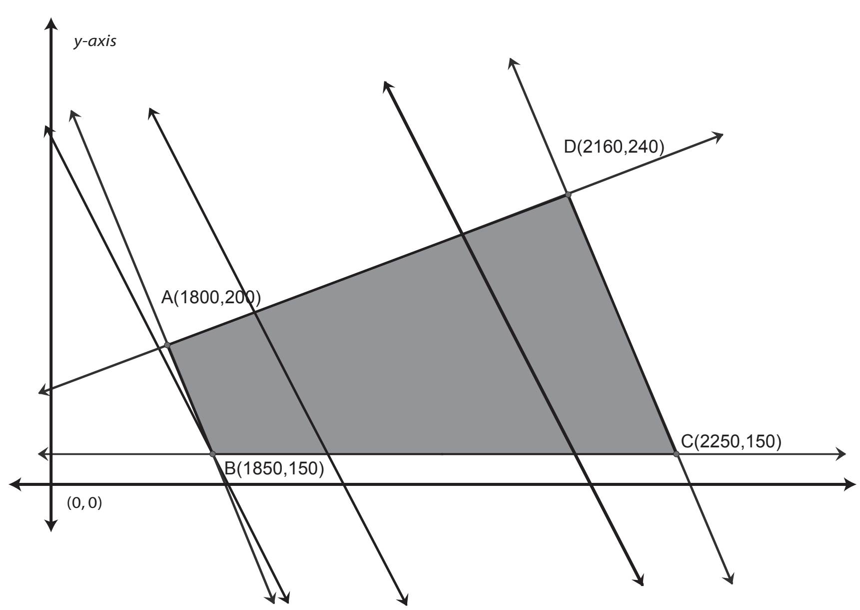
- How would you determine x- and y-values that will satisfy the equation $580x + 600y = C$?

  We could solve for y in terms of x to get $y = \frac{C - 580x}{600}$. *Then we could use the calculator table or graph.*

- Experiment with a few different C values close to the value that minimizes the cost. Graph the resulting cost equations. What do you notice about the graphs?

  The following have been graphed with the original functions:

$y = \frac{1{,}163{,}000 - 580x}{600}$ $y = \frac{1{,}192{,}000 - 580x}{600}$ $y = \frac{1{,}308{,}000 - 580x}{600}$



The graphs of the cost equations are parallel lines. When the cost is the minimum, \$1,163,000, the line goes through (1,850,150).

- Can you make a conjecture about the location of the vertex that minimizes cost?

  Yes. Draw a line using a fixed cost. Move it parallel to itself from right to left across the feasible region. The last vertex it passes through will minimize the cost.

- Your calculator graph looks like the line corresponding to the minimal cost is the same as the boundary line $y = 2000 - x$. Is that true?

  No. The slope of the minimal cost equation is $m = \frac{-580}{600} = -0.9\overline{6}$*. That is so close to the slope of the boundary line, $m = -1$, it is hard to distinguish between the lines.*

- What would you conjecture to be true about the location of the vertex that will maximize the weekly revenue?

  Draw a line representing a particular weekly revenue, R, given by $1{,}000x + 10y = R$*. Move it parallel to itself from left to right across the feasible region. The last vertex the line passes through will maximize the revenue.*

The Mild and Wild Amusement Park

Three friends, Travis, Kaitlyn, and Karsyn, spent the day at Mild and Wild Amusement Park, which features rides classified as Mild, Wild, or Super Wild.

The park had 2 ticket purchase options.

> Option One: Pay a $5 admission fee and buy a ticket at regular price for each ride individually.
>
> Option Two: Pay a $5 admission fee and buy a ticket book that includes 8 tickets for each of the 3 different types of rides at a 20% discount per ticket.

The 3 friends chose to pay with Option One. They paid the admission fee plus the regular ticket cost for each ride they chose.

By the end of the day, Travis had ridden on 4 Mild rides, 8 Wild rides, and 8 Super Wild rides for a total ticket cost of $26. Kaitlyn had ridden on 8 Mild rides, 7 Wild rides, and 5 Super Wild rides for a total ticket cost of $24.25. Karsyn had ridden on 7 Mild rides, 6 Wild rides, and 4 Super Wild rides for a total ticket cost of $20.50.

1. Determine the ticket price for each type of ride—Mild, Wild, and Super Wild. Solve an algebraic system for this situation using matrices and technology.

2. Determine the amount each person would spend if he or she had chosen Option Two and ridden the same combination of rides. Explain which method of payment would have been best for each person for their day at the amusement park.

Notes

CCSS Content Task

(A-REI) **Solve systems of equations**

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.

9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Standards for Mathematical Practice
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.

Scaffolding Questions:

- What information is known in the problem?
- What are the unknowns in the problem?
- How will you organize the known information in a matrix?
- How will you then organize the unknowns and the total amount spent by the friends in matrices?
- What matrix equation can you now write?
- How do you solve the matrix equation?
- How will you use your calculator to solve the problem?

Sample Solutions:

1. We know the number of rides each person took and the total amount they paid for tickets. We need to determine the price of a ticket for each type of ride.

   We can write a matrix equation for this situation. The 3 by 3 coefficient matrix, *A*, will represent the 3 friends (rows) and the number of each of the 3 types of ride he or she took (columns). The ticket price matrix, *X*, will be a 3 by 1 matrix.

   Let m = the ticket price for a Mild ride

   w = the ticket price for a Wild ride

   s = the ticket price for a Super Wild ride.

   The constant matrix will be a 3 by 1 matrix, *P*, of the total each person paid for tickets.

$$A=\begin{bmatrix}4 & 8 & 8\\ 8 & 7 & 5\\ 7 & 6 & 4\end{bmatrix},\quad X=\begin{bmatrix}m\\ w\\ s\end{bmatrix},\ P=\begin{bmatrix}26.00\\ 24.25\\ 20.50\end{bmatrix}$$

The matrix equation to solve is $AX = P$. To do this we enter A and P as matrices in the calculator and get the solution by computing A^{-1} and the product $A^{-1}P$. This gives us the ticket price matrix X, since

$$A^{-1}AX = A^{-1}P$$
$$X = A^{-1}P$$
$$X = A^{-1}P$$
$$X = \begin{bmatrix} m \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.25 \\ 1.50 \end{bmatrix}$$

The regular ticket prices for Mild, Wild, and Super Wild rides are \$1, \$1.25, and \$1.50, respectively.

2. For Option Two, we need to know the discounted price of the tickets, which is 80% of the ticket price matrix, X.

$$D = 0.80X = 0.80\begin{bmatrix} 1.00 \\ 1.25 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 0.80 \\ 1.00 \\ 1.20 \end{bmatrix}$$

For Option Two, each person would pay the admission fee and the price of the ticket book, which would be 5 + 8(0.80 + 1.00 +1.20) = 29 dollars.

The table below compares how the 3 friends would fare with each option.

	Spent with Option One: admission fee plus ticket cost	Cost of Option Two	Better buy
Travis	\$5.00 + \$26.00 = \$31.00	\$29.00	Option Two
Kaitlyn	\$5.00 + \$24.25 = \$29.25	\$29.00	Option Two
Karsyn	\$5.00 + \$20.50 = \$25.50	\$29.00	Option One

Travis and Kaitlyn would have gotten a better deal with Option Two, while Karsyn was better off with Option One.

Extension Questions:

- What linear system corresponds to the matrix equation you solved in this problem? How can you obtain that system from the matrix equation?

  To get the corresponding linear system, we multiply each row in the coefficient matrix and the ticket price matrix, term by term, and sum the products. We set that equal to the corresponding entry in the constant matrix. The system is

$$
\begin{aligned}
4m + 8w + 8s &= 26.00 \\
8m + 7w + 5s &= 24.25 \\
7m + 6w + 4s &= 20.50
\end{aligned}
$$

- What method would you use to solve this system? How would the work to do that compare with the matrix solution?

  We would solve the system using the Linear Combination Method. This is not as efficient as solving the system as a matrix equation.

- Suppose the amusement park had a fourth type of ride, called Colossal Wild. In addition to the other rides, Travis rode 2 Colossal Wild rides and spent \$30. Kaitlyn rode 3 Colossal Wild rides and spent \$30.25. Karsyn rode 1 Colossal Wild ride and spent \$22.50. Suppose we do not have the information found for the original problem. Would you be able to write and solve a matrix equation for this new situation?

  No. The coefficient matrix, A, would have 3 rows (one for each of the friends) and 4 columns (one for each type of ride). The ticket price matrix, X, would have 4 rows and 1 column. You cannot solve the equation AX = P by multiplying both sides of the equation by A^{-1}*. A is not a square matrix, so its inverse does not exist.*

- What would the corresponding linear system look like?

  It would be three equations in four unknowns, which we cannot solve.

  But we could figure out the ticket price for a Colossal Wild ride ticket if we knew the price of the others.

- In groups, create a situation involving four unknowns that will generate a 4 by 4 system. Groups will exchange situations, solve them, and share results.

  Answers will vary.

Task 1:

We found that the price of a mild ride was \$1, the price of a wild ride was \$1.25, and the price of a super wild ride was \$1.50. We determined this by setting up (1) algebraic system of equations, then entering the information into (2) 3 different matrices: a coeffecient matrix, a variable matrix, and an answer matrix.

(1)
$$4m + 8w + 8s = \$26.00$$
$$8m + 7w + 5s = \$24.25$$
$$7m + 6w + 4s = \$20.50$$

(2)
$$\begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} m \\ w \\ s \end{bmatrix} = \begin{bmatrix} \$26.00 \\ \$24.25 \\ \$20.50 \end{bmatrix}$$

We then set up this equation

(3)
$$\begin{bmatrix} M \\ W \\ S \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} \$26.00 \\ \$24.25 \\ \$20.50 \end{bmatrix}$$

We solved this on the calculator and found M to be \$1, W to be \$1.25, and S to be \$1.50.

→

Task 2:

We needed to find out if the three friends would have saved money if they had chosen to pay with option **two**. We found this using the following methods.

First, we needed to determine the price of each ticket with a 20% discount. We did this by multiplying .2 by the cost of each ticket, then subtracting the product from the original price.

M	W	S
$.2 \times \$1.00 = .20$	$.2 \times \$1.25 = .25$	$.2 \times \$1.50 = .30$
$\$1.00 - .20 = \boxed{.80}$	$\$1.25 - .25 = \boxed{\$1.00}$	$\$1.50 - .30 = \boxed{\$1.20}$

Then we needed to find the total cost of the package in option two. We found this by multiplying 8 by \$0.80, \$1.00, and \$1.50, then adding the products togeather. The sum was \$24.00. We then added the \$5 admission.

If they had bought option two, each person would have payed \$29.

In buying option one, they each payed the entrance fee of \$5.00 plus the total cost of their tickets.

Travis: \$26.00 + \$5.00 = \$31.00
Kaitlyn: \$24.25 + \$5.00 = \$29.25
Karsyn: \$20.50 + \$5.00 = \$25.50

By subtracting the amount that they would have spent in buying option one from the amount that they spent by buying option two, we discovered that they would have saved the following amounts by buying option one.

Travis: \$29.00 − \$31.00 = \$2.00
Kaitlyn: \$29.00 − \$29.25 = \$-.25
Karsyn: \$29.00 − \$25.50 = \$4.50

Therefore, Travis would have saved \$2.00. Kaitlyn would have 25¢, and Karsyn would have lost \$4.50.

Weather Woes

Storm E. Freeze has been fascinated with weather phenomena since childhood. After she graduates from college, Storm would like to become a meteorologist for a national weather syndicate, and she knows that she must be able to convert Celsius temperatures to Fahrenheit temperatures with ease. In addition, her job as a meteorologist will require that she be able to explain how the formulas for each are related and describe the conversions verbally, graphically, and symbolically.

She knows there is a linear relationship between the Celsius measure and the Fahrenheit measure. She has recorded the following measures:

Celsius temperature °*C*	Fahrenheit temperature °*F*
5	41
14	57.2

You have agreed to help her with her math project involving temperature conversion and inverses.

1. Determine a formula to express *F* in terms of *C*.

2. Determine a formula to express *C* in terms of *F*.

3. Explain algebraically why these are inverse functions.

4. Graph the two functions on the same set of axes. Describe the graphs, their domains, and their ranges. How do the graphs help determine if the functions are inverses? Explain the meaning of the point of intersection of the two graphs.

Notes

CCSS Content Task

(8.EE) **Analyze and solve linear equations and pairs of simultaneous linear equations.**

8. Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

(8.F) **Use functions to model relationships between quantities.**

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

(A-CED) **Create equations that describe numbers or relationships**

2. Create equations in two or more variables to represent relationships between quantities; graph equations on

Scaffolding Questions:

- What must you know to determine a linear function rule?
- What information can you get from the table?
- How can you determine whether two functions are inverses of one another?
- Graphs that are inverses of one another have a special property. What is that property?

Sample Solutions:

1. Use the table to determine the slope of the linear function.

Celsius temperature °*C*	**Fahrenheit temperature °*F***
5	41
14	57.2

$$\frac{\text{change in } F}{\text{change in } C} = \frac{57.2 - 41}{14 - 9} = \frac{16.2}{9} = 1.8$$

$F = 1.8C + b$

Use one of the points to determine the value of *b*.

$41 = 1.8(5) + b$

$b = 32$

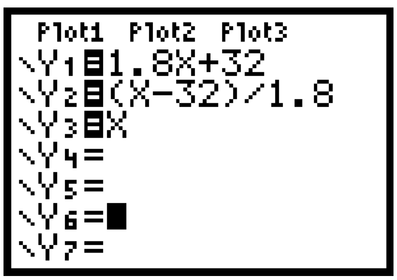
$F = 1.8C + 32$

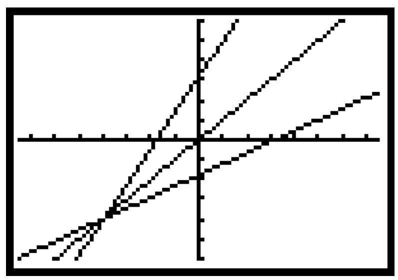
2. Rewrite the formula to solve for C in terms of F.

$$F = 1.8C + 32$$
$$F - 32 = 1.8C$$
$$C = \frac{F - 32}{1.8}$$

3. It makes sense that these formulas would be inverses of each other if, when you convert from one temperature scale to another and back again, you arrive at the original temperature.

4. Enter the graphs of the two functions into a graphing calculator. The line $y = x$ is also graphed to show the relationship between the functions.





Both functions are linear functions. For every point (a, b) on the graph of the original function, you can find the point (b, a) on the graph of the inverse function. The graphs of the function and its inverse are symmetric to each other with respect to the line $y = x$. Graphs of the function and its inverse are reflections of each other over the line $y = x$.

This relationship can also be shown algebraicially.

$$C = 1.8F + 32$$
$$F = \frac{C - 32}{1.8}$$

coordinate axes with labels and scales.

(A-CED) **Create equations that describe numbers or relationships**

4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.**

(A-REI) **Represent and solve equations and inequalities graphically**

11. Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(F-IF) **Interpret functions that arise in applications in terms of the context**

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Notes

*Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

(F-IF) **Analyze functions using different representations**

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

(F-BF) **Build a function that models a relationship between two quantities**

1. Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

(F-BF) **Build new functions from existing functions**

4. Find inverse functions.

a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, f(x) =2 x3 or f(x) = (x+1)/(x–1) for x ≠ 1.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

(F-LE) **Construct and compare linear, quadratic, and exponential models and solve problems**

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).*

Standards for Mathematical Practice
4. Model with mathematics.
8. Look for and express regularity in repeated reasoning.

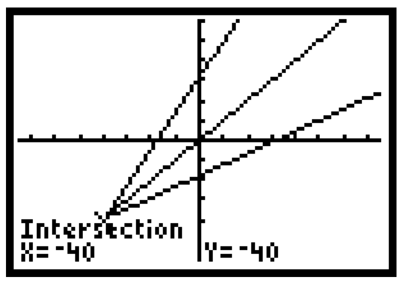
We can check to see if the functions are inverses by substituting for *F* in the first rule.

$$C = 1.8\left(\frac{C-32}{1.8}\right)+32$$
$$C = C-32+32 = C$$

The functions are inverses of each other.

The horizontal axis and the vertical axis represent degree measures. The domain of one of the functions is equal to the range of its inverse. The range of the function is equal to the domain of the inverse. Each number in the domain corresponds to a unique number in its range and vice versa.

The meaning of the intersection point is that if a measure is -40 degrees in Celsius it is equal to -40 degrees in Fahrenheit.



Extension Questions:

- The functions modeled in the problem were from only one family of functions. Consider all of the other families of functions,

  $$y = x, y = x^2, y = |x|, y = \sqrt{x}, y = a^x, y = \log_a x, \text{ and } y = \frac{1}{x}$$

  and determine any functions that are inverses of each other.

An inverse function has the characteristic of "undoing" the operations of the original function. The inverse of the function y = x^2 *is the square root function* $y = \sqrt{x}$ *if you restrict the domain and range of the function to be all non-negative real numbers.*

The inverse of a power function such as y = 2^x *is y = log*2^x.

The function y = x is its own inverse. The function $y = \frac{1}{x}$ *is its own inverse.*